Table 1.6. Nan's payoff matrix in Four Ways

	G	W	C
G	$-\delta - \frac{1}{2}\tau$	0	0
W	$-\tau$	$-\epsilon - \frac{1}{2}\tau$	$-\tau$
C	$-\tau$	0	$-\delta - \frac{1}{2}\tau$

Table 1.7. San's payoff matrix in Four Ways

	G	W	C
G	$-\delta - \frac{1}{2}\tau$	$-\tau$	$-\tau$
W	0	$-\epsilon - \frac{1}{2}\tau$	0
C	0	$-\tau$	$-\delta - \frac{1}{2}\tau$

1.4. Four Ways: a motorist's trilemma

Nan and San's dilemma becomes even more intriguing if we allow a third strategy, denoted by C, in which each player's action is contingent upon that of the other. A player who adopts C will select G if the other player selects W, but she will select W if the other player selects G. Let us suppose that, if Nan is a C-strategist, then the first thing she does when she arrives at the junction is to wave San on; but if San replies by waving Nan on, then immediately Nan puts down her foot and drives away. If, on the other hand, San replies by hitting the gas, then Nan waits until San has traversed the junction. But what happens if San is also a C-strategist? As soon as they reach the junction, Nan and San both wave at one another. Nan interprets San's wave to mean that San wants to wait, so Nan drives forward; San interprets Nan's wave to mean that Nan wants to wait, so San also drives forward; and the result is the same as if both had selected strategy G. Thus if a G-strategist can be described as selfish and a W-strategist as an altruist, then a C-strategist could perhaps be described as an impatient altruist.

For the sake of simplicity, let us assume that the game is symmetric, i.e., $\tau_1 = \tau_2$, and denote the common value of these two parameters by τ. Then Nan and San's payoff matrices A and B, respectively, are as shown in Tables 1.6 and 1.7. As always, the rows correspond to strategies of Player 1 (Nan), and the columns correspond to strategies of Player 2 (San); thus the entry in row i and column j is the payoff, to the player whose payoffs are stored in the matrix, if Player 1 selects strategy i and Player 2 selects strategy j. Because the game is symmetric, B is just the transpose of A. To distinguish this game from Crossroads, we will refer to it as Four Ways.

If the drivers are so slow that $\tau > 2\delta$ or $\sigma > 1$, where

$$\sigma \;=\; \tau/2\delta, \tag{1.27}$$

then their best strategy is to hit the gas, because G dominates C and strictly dominates W for Nan, from Table 1.6; and similarly for San, from Table 1.7. Thus G is a (weakly) dominant strategy for both players: neither has an incentive to depart from it, which makes strategy combination GG a Nash equilibrium. Furthermore, GG is the only Nash equilibrium when $\sigma > 1$ (Exercise 1.3), and so we do not hesitate to regard it as the solution of the game: when there is only one Nash equilibrium, there is no indeterminacy to resolve.[8]

The game becomes interesting, however, when $\tau < 2\delta$ or $\sigma < 1$, which we assume for the rest of this section. As in Crossroads, no pure strategy is now dominant. We therefore consider mixed strategies. If Nan selects pure strategy G with probability u_1 and pure strategy W with probability u_2, then we shall say that Nan selects strategy u, where $u = (u_1, u_2)$ is a 2-dimensional row vector. Thus Nan selects pure strategy C with probability $1 - u_1 - u_2$, where

$$0 \le u_1 \le 1, \qquad 0 \le u_2 \le 1, \qquad 0 \le u_1 + u_2 \le 1. \tag{1.28a}$$

So Nan's strategies correspond to points of a closed triangle in 2-dimensional space. Similarly, if San selects G with probability v_1 and W with probability v_2, then we shall say that San selects strategy v, where $v = (v_1, v_2)$ is also a 2-dimensional vector; and because San selects C with probability $1 - v_1 - v_2$, we have

$$0 \le v_1 \le 1, \qquad 0 \le v_2 \le 1, \qquad 0 \le v_1 + v_2 \le 1. \tag{1.28b}$$

Subsequently, we shall use Δ to denote the closed triangle in 2-dimensional space defined EITHER as the set of all points that satisfy

[8] Even if there were more than one Nash equilibrium, there would be no indeterminacy if all combinations of Nash-equilibrium strategies yielded the same payoffs. This equivalence holds in general only for zero-sum games; see, for example, Owen [**173**] or Wang [**233**]. For an example of a zero-sum game, see Exercise 1.33.

(1.28a) OR as the set of all points that satisfy(1.28b); the sets are identical, because this triangle exists independently of whether we use u or v to label a point in it. If Nan selects $u \in \Delta$ and San selects $v \in \Delta$, then we shall say that they jointly select strategy combination (u, v), where $(u, v) = (u_1, u_2, v_1, v_2)$ is a 4-dimensional vector.

The sample space of N, Nan's choice of pure strategy, is now $\{G, W, C\}$ instead of $\{G, W\}$; $\mathrm{Prob}(N = G) = u_1$, $\mathrm{Prob}(N = W) = u_2$ and $\mathrm{Prob}(N = C) = 1 - u_1 - u_2$. San's choice of pure strategy, S, has the same sample space, but with $\mathrm{Prob}(S = G) = v_1$, $\mathrm{Prob}(S = W) = v_2$ and $\mathrm{Prob}(S = C) = 1 - v_1 - v_2$. The payoff to Nan, F_1, now has sample space $\{-\delta - \frac{1}{2}\tau, 0, -\tau, -\epsilon - \frac{1}{2}\tau\}$; and if strategies are still chosen independently, then $\mathrm{Prob}(F_1 = -\delta - \tau/2) = \mathrm{Prob}(N = G, S = G \text{ or } N = C, S = C) = \mathrm{Prob}(N = G, S = G) + \mathrm{Prob}(N = C, S = C) = \mathrm{Prob}(N = G) \cdot \mathrm{Prob}(S = G) + \mathrm{Prob}(N = C) \cdot \mathrm{Prob}(S = C) = u_1 v_1 + (1 - u_1 - u_2)(1 - v_1 - v_2)$. Similarly, $\mathrm{Prob}(F_1 = 0) = u_1 v_2 + u_1(1 - v_1 - v_2) + (1 - u_1 - u_2)v_2$, $\mathrm{Prob}(F_1 = -\tau) = u_2 v_1 + u_2(1 - v_1 - v_2) + (1 - u_1 - u_2)v_1$ and $\mathrm{Prob}(F_1 = -\epsilon - \tau/2) = u_2 v_2$. Thus Nan's reward from the mixed strategy combination (u, v) is $f_1(u, v) = \mathrm{E}[F_1] = -\big(\delta + \frac{1}{2}\tau\big) \cdot \mathrm{Prob}\big(F_1 = -\delta - \frac{1}{2}\tau\big) + 0 \cdot \mathrm{Prob}\big(F_1 = 0\big) - \tau \cdot \mathrm{Prob}\big(F_1 = -\tau\big) - \big(\epsilon + \frac{1}{2}\tau\big) \cdot \mathrm{Prob}\big(F_1 = -\epsilon - \frac{1}{2}\tau\big)$ or, after simplification,

$$
\begin{aligned}
f_1(u, v) \;=\; &-\big(2\delta v_1 + \big\{\delta + \tfrac{1}{2}\tau\big\}\{v_2 - 1\}\big)u_1 \\
&- \big(\big\{\delta - \tfrac{1}{2}\tau\big\}\{v_1 - 1\} + \{\delta + \epsilon\}v_2\big)u_2 \\
&+ \big(\delta - \tfrac{1}{2}\tau\big)v_1 + \big(\delta + \tfrac{1}{2}\tau\big)(v_2 - 1).
\end{aligned} \tag{1.29}
$$

Similarly, San's reward from the strategy combination (u, v) is

$$
\begin{aligned}
f_2(u, v) = &-\big(2\delta u_1 + \big\{\delta + \tfrac{1}{2}\tau\big\}\{u_2 - 1\}\big)v_1 \\
&- \big(\big\{\delta - \tfrac{1}{2}\tau\big\}\{u_1 - 1\} + \{\delta + \epsilon\}u_2\big)v_2 \\
&+ \big(\delta - \tfrac{1}{2}\tau\big)u_1 + \big(\delta + \tfrac{1}{2}\tau\big)(u_2 - 1).
\end{aligned} \tag{1.30}
$$

Note that, by virtue of symmetry,

$$
f_2(u, v) = f_1(v, u) \tag{1.31}
$$

for all u and v satisfying (1.28). Note also that (1.29) and (1.30) are special cases of (1.15).

Although u and v are now vectors, as opposed to scalars, everything we have said about rational reaction sets and Nash equilibria with respect to Crossroads remains true for Four Ways, provided only that we replace $0 \le u \le 1$ by $u \in \Delta$ and $0 \le v \le 1$ by $v \in \Delta$ (and therefore also $0 \le \overline{u} \le 1$ by $\overline{u} \in \Delta$ and $0 \le \overline{v} \le 1$ by $\overline{v} \in \Delta$). Thus the players' rational reaction sets in Four Ways are defined by

$$R_1 = \left\{(u, v) \mid u, v \in \Delta,\ f_1(u, v) = \max_{\overline{u}} f_1(\overline{u}, v)\right\} \tag{1.32a}$$

$$R_2 = \left\{(u, v) \mid u, v \in \Delta,\ f_2(u, v) = \max_{\overline{v}} f_2(u, \overline{v})\right\}, \tag{1.32b}$$

but the set of all Nash equilibria is still $R_1 \cap R_2$. On the other hand, because the rational reaction sets now lie in a 4-dimensional space, as opposed to a 2-dimensional space, we cannot locate the Nash equilibria by drawing diagrams equivalent to Figures 1.3-1.5. Instead, we proceed as follows. We first define dimensionless parameters

$$\gamma = \frac{\epsilon}{\delta},\ \alpha = \frac{(\sigma+\gamma)(\sigma+1)}{1+2\gamma+\sigma^2},\ \beta = \frac{(1-\sigma)^2}{1+2\gamma+\sigma^2},\ \omega = \frac{2\sigma}{1+\sigma} \tag{1.33}$$

and

$$\theta = \frac{2\epsilon+\tau}{2\epsilon+2\delta} = \frac{\sigma+\gamma}{1+\gamma} \tag{1.34}$$

where σ is defined by (1.27). In view of (1.1), α, β, γ, σ, θ and ω all lie between 0 and 1. If the coefficients of u_1 and u_2 in (1.29) are both negative, then clearly $f_1(u, v)$ is maximized by selecting $u_1 = 0$ and $u_2 = 0$, or $u = (0, 0)$; moreover, $(0, 0)$ is the only maximizing strategy for Player 1. If these coefficients are merely nonpositive, then there will be more than one maximizing strategy; nevertheless, $u = (0, 0)$ will continue to be one of them. But the coefficient of u_1 in (1.29) is nonpositive when the point (v_1, v_2) lies on or above the line in 2-dimensional space that joins the point $(\sigma/\omega, 0)$ to the point $(0, 1)$; whereas the coefficient of u_2 in (1.29) is nonpositive when the point (v_1, v_2) lies on or above the line that joins the point $(1, 0)$ to the point $(0, 1-\theta)$. Thus the coefficients of u_1 and u_2 in (1.29) are both nonpositive when the point (v_1, v_2) lies in that part of Δ which corresponds to (the interior or boundary of) the triangle marked C in Figure 1.6. Let us denote by $v^C = (v_1^C, v_2^C)$ any strategy for San that corresponds to a point in C. Then what we have shown is that all 4-dimensional vectors of the form $\left(0, 0, v_1^C, v_2^C\right)$ must lie in R_1.

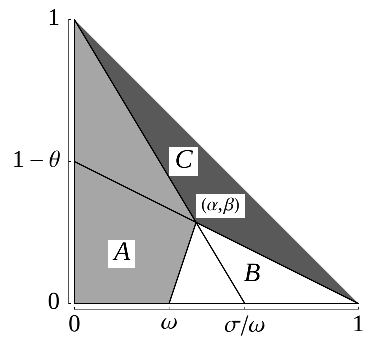

Figure 1.6. Subsets A, B and C of Δ defined by (1.28).

Extending our notation in an obvious way, let us denote by $v^A = (v_1^A, v_2^A)$ any strategy for San that corresponds to a point in A, by $v^{AC} = (v_1^{AC}, v_2^{AC})$ any strategy for San that corresponds to a point lying in both A and C, and so on. Then, by considering the various cases in which the coefficient of u_1 or the coefficient of u_2 or both in (1.29) are nonpositive, nonnegative or zero, it is readily shown that all strategy combinations in Table 1.8 must lie in Nan's rational reaction set, R_1; see Exercise 1.5. Furthermore, if we repeat the analysis for f_2 and San (as opposed to f_1 and Nan), and if we denote by $u^A = (u_1^A, u_2^A)$ any strategy for Nan that corresponds to a point in A, by $u^{AC} = (u_1^{AC}, u_2^{AC})$ any strategy for Nan that corresponds to a point in both A and C, and so on, then we readily find that all strategy combinations in Table 1.9 must lie in San's rational reaction set, R_2. Indeed, in view of symmetry condition (1.31), it is hardly necessary to repeat the analysis.

A strategy combination is a Nash equilibrium if, and only if, it appears both in Table 1.8 and in Table 1.9. Therefore, to find all Nash equilibria, we must match strategy combinations from Table 1.8 with strategy combinations from Table 1.9 in every possible way. For example, consider the first row of Table 1.8. It does not match the first, fourth or sixth row of Table 1.9 because $(1, 0)$ does not lie in A. It does not match the last row of Table 1.9, even for $(v_1, v_2) \in A$, because $\alpha < 1$ (or because $\beta > 0$). Because $(1, 0)$ lies in B and

Table 1.8. R_1 for Four Ways.

u_1	u_2	v_1	v_2	CONSTRAINTS
1	0	v_1^A	v_2^A	
0	1	v_1^B	v_2^B	
0	0	v_1^C	v_2^C	
u_1	0	v_1^{AC}	v_2^{AC}	$0 \le u_1 \le 1$
0	u_2	v_1^{BC}	v_2^{BC}	$0 \le u_2 \le 1$
u_1	u_2	v_1^{AB}	v_2^{AB}	$u \in \Delta,\ u_1 + u_2 = 1$
u_1	u_2	α	β	$u \in \Delta$

Table 1.9. R_2 for Four Ways.

u_1	u_2	v_1	v_2	CONSTRAINTS
u_1^A	u_2^A	1	0	
u_1^B	u_2^B	0	1	
u_1^C	u_2^C	0	0	
u_1^{AC}	u_2^{AC}	v_1	0	$0 \le v_1 \le 1$
u_1^{BC}	u_2^{BC}	0	v_2	$0 \le v_2 \le 1$
u_1^{AB}	u_2^{AB}	v_1	v_2	$v \in \Delta,\ v_1 + v_2 = 1$
α	β	v_1	v_2	$v \in \Delta$

Table 1.10. Nash equilibria for Four Ways.

u_1	u_2	v_1	v_2	CONSTRAINTS
1	0	0	1	
0	1	1	0	
1	0	0	0	
0	0	1	0	
1	0	0	v_2	$0 \le v_2 < 1$
0	u_2	1	0	$0 \le u_2 < 1$
0	1	v_1	0	$\omega \le v_1 < 1$
u_1	0	0	1	$\omega \le u_1 < 1$
α	β	α	β	

$(0, 1)$ lies in A, however, we can match the first row of Table 1.8 with the second row of Table 1.9, and so $(1, 0, 0, 1)$ is a Nash equilibrium. Likewise, because $(1, 0)$ lies in C and $(0, 0)$ in A, we can match the first row of Table 1.8 with the third row of Table 1.9, so that $(1, 0, 0, 0)$ is a Nash equilibrium. Finally, we can match the first row of Table 1.8 with the fifth row of Table 1.9 to deduce that $(1, 0, 0, v_2)$ is a Nash-equilibrium strategy combination when $0 \leq v_2 < 1$, because then $(0, v_2)$ lies in A. The Nash equilibria we have found in this way are recorded in rows 1, 3 and 5 of Table 1.10.

Repeating the analysis for the remaining six rows of Table 1.8, we obtain (Exercise 1.6) an exhaustive list of Nash-equilibrium strategy combinations. They are recorded in Table 1.10. The first four rows of this table correspond to equilibria in pure strategies: rows 1 and 2 to equilibria in which one player selects G and the other W, rows 3 and 4 to equilibria in which one player selects G and the other C. The remaining five rows correspond to equilibria in mixed strategies. We see that, although rows 1-4 and 9 of the table correspond to isolated equilibria, there are infinitely many equilibria of the other types. If you thought that having three equilibria to choose from in Crossroads was bad enough, then I wonder what are you thinking now. Which, if any, of all these infinitely many equilibria do we regard as the solution of Four Ways?

Good question! Perhaps you would like to mull it over, at least until Chapter 2. Meanwhile, do Exercise 1.29.

1.5. Store Wars: a continuous game of prices

Although it is always reasonable to suppose that decision makers have only a finite number of pure strategies, when the number is large it is often convenient to imagine instead that the strategies form a continuum. Suppose, for example, that the price of some item could reasonably lie anywhere between five and ten dollars. Then if a cent is the smallest unit of currency, and if selecting a strategy corresponds to setting the price of the item, then the decision maker has a finite total of 501 pure strategies. Because this number is large, however, it may be preferable to suppose that the price in dollars can take any value between 5 and 10 (and round to two decimal places). Then rewards are calculated directly, i.e., without the intermediate step of calculating payoff matrices; and the game is said to be *continuous*, to distinguish it from matrix games like Crossroads, Four Ways and the Hawk-Dove game. The definition of Nash equilibrium is not in the